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Coherent states for the hydrogen atom: discrete and continuous spectra

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Abstract

We construct the systems of generalized coherent states for the discrete and continuous spectra of the hydrogen atom. These systems are expressed in elementary functions and are invariant under the $SO(3, 2)$ (discrete spectrum) and $SO(4, 1)$ (continuous spectrum) subgroups of the dynamical symmetry group $SO(4, 2)$ of the hydrogen atom. Both systems of coherent states are particular cases of the kernel of integral operator which intertwines irreducible representations of the $SO(4, 2)$ group.

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1. Introduction

The problem of constructing the generalized coherent states (CS) for the hydrogen atom (HA) was first formulated by Schrödinger in 1926 simultaneously with the construction of the CS for the harmonic oscillator (HO) (see historical remarks in [1]). Since then, there have been attempts to solve this problem on the basis of the Kustaanheimo–Stiefel transformation connecting 4D HO and 3D HA [2, 3]. Within this approach, the CS for 4D HO are constructed, whereas the CS for 3D HA are obtained from the above CS either using constraints imposed on a set of parameters [2] or by integration over an additional variable [3]. However, the packets obtained in [3] spread with time; even within the time when the packets preserve their shape, they can be considered as simulating the Kepler motion for large quantum numbers only and, as it was shown in [4], when restricted to a plane. The same statements are also valid for the states constructed in [2].

It is also possible (as it was suggested in [5]) to start from the existence of integrals of motion. Due to the dynamical symmetry group $SO(4, 2)$ of the HA [6], this idea naturally leads to the construction of CS by the Barut–Girardello method, as it was performed in [7]. Such states evolve consistently under the pseudo-Hamiltonian $(-2H)^{-1/2}$; however, during evolution in real time they spread and cannot be expressed in a closed form in configuration space. Much progress has been achieved in constructing the CS for the radial Schrödinger equation for the HA [8, 9]. These states are expressed in elementary functions of r, are $SO(2, 1)$-invariant and minimize the uncertainty relation for the suitably defined operators X and P.

Thus, the attempts to establish the quantum–classical correspondence for the 3D HA, using the HO as a model, failed. In fact, an extremely simple form of this correspondence for the HO is a consequence of the fact that its energy levels are equidistant; as a result, such simple correspondence cannot exist for the HA. This circumstance has been recently mentioned in [10]; however, this reasoning can be found even in one of the Bohr's fundamental papers on quantum mechanics [11] (with reference to Heisenberg and Darwin).

Therefore, in our opinion, the formulation of quantum–classical correspondence for the HA should be based on mathematical principles rather than on the dynamical considerations. Recently, two approaches have been forwarded. The first approach suggested by Klauder [12] is based on his continuous-representation theory [13] and later was developed in a number of papers (see [14] and references therein). This approach implies that the CS system should obey the following requirements: (i) the dependence on the parameters is continuous, (ii) the resolution of the identity takes place and (iii) the Hamiltonian yields the evolution in parameter space. The disadvantages of this approach include the infiniteness of the evolution in the parameter space, which is inconsistent with periodicity of the corresponding classical motion, and the absence of a closed-form expression for the CS in configuration space.

Another approach goes back to the Mostowski's paper [15] and is based on the Perelomov's method [16] of constructing the CS for an arbitrary Lie group; any dynamical considerations are avoided. In the context of this approach, we constructed a CS system for the discrete spectrum of HA [17]; in particular, this system corresponds to the $SO(3, 2)/SO(3) \otimes SO(2)$ coset space and is expressed in the closed form in configuration space.

In this paper, we present the construction of this CS system in more detail and show that the difficulties associated with minimization of the uncertainty relation for the HA CS noted in [18] can be overcome if one uses correct definitions of both. We also show that the CS system constructed in [17] admits a natural generalization to the continuous spectrum of HA; in this case, the symmetry group is $SO(4, 1)$. Previously, a CS system for the $Sp(1, \mathbb{R})/U(1)$ space and for a continuous series of the $Sp(1, \mathbb{R})$ group representations was constructed in a different way [19]. Discussion of some other properties of CS for the discrete spectrum of HA and their comparison with the HO CS properties may be found in [17]; it is noteworthy that notations used here coincide only partly with those in [17]. Concerning the physical meaning of the symmetric space which corresponds to the CS system for the HA discrete spectrum, see [39].

The plan of the present paper is the following. In section 2.1 we consider the Kustaanheimo–Stiefel transformation and its relation to wave functions of the discrete spectrum of HA. Since the Kustaanheimo–Stiefel transformation and its generalizations have been considered in many papers (see, e.g. [2, 20–22] and references therein), we present here the needed information only and mostly will follow the approach outlined in [2, 23, 24]. In section 2.2 we give the group-theoretical treatment of the HA discrete spectrum, mostly following [24, 25]. In section 3.1 we consider the $SO(3, 2)$ group and the corresponding $SO(3, 2)/(SO(3) \otimes SO(2))$ symmetric space. In section 3.2 a CS system for the discrete spectrum is constructed; CS for the 1D HA constructed in [9] are the particular case of our CS. In section 3.3 we show that this system is a $SO(3, 2)$-invariant system of the Perelomov's CS. In section 3.4 we show that this CS system minimizes the so-called Robertson inequality for 4D coordinates and momenta; this inequality is a generalization of the Heisenberg uncertainty relation to the case of n variables. In this respect, the CS system constructed is similar to the usual CS for the Heisenberg–Weyl group. In section 4 the continuous spectrum is considered. In section 4.1 we consider the $SO(4, 1)$ group and its action on $\mathbb{R}^3$ (concerning the conformal action of orthogonal groups over Euclidean and pseudoeuclidean spaces see also [26]). In section 4.2 the wavefunctions of the continuous spectrum of HA are considered. In section 4.3 we follow the ideas of [27] concerning the Mellin transform of the confluent hypergeometric functions to construct the CS system for the continuous spectrum of HA; this system is similar to that constructed in section 3.2. In section 5 it is shown that the CS systems for the discrete and continuous spectra of HA are particular cases of a function which intertwines the different irreducible representations of the $SO(4, 2)$ group (for more details on this group and its representations, see [28, 29] and references therein). We also establish a relation between the results obtained above and that reported recently [30] concerning representation of the HA wavefunctions in terms of the classical motion.

2. Preliminaries

2.1. The Kustaanheimo–Stiefel transformation

Let us introduce the four-vector

$$n_{\boldsymbol{x}}^{\mu} = (r, \boldsymbol{x}) \qquad n_{\boldsymbol{x}} \cdot n_{\boldsymbol{x}} = 0 \quad n_{\boldsymbol{x}}^{0} \geqslant 0 \quad \mu, \nu, \ldots = 0, \ldots, 3.$$

Denote the Cartesian coordinates in $\mathbb{R}^4$ as $\xi_\alpha, \eta_\alpha, \alpha, \beta = 1, 2$. We also need two other coordinate systems in $\mathbb{R}^4$: $\xi, \eta, \varphi_\xi, \varphi_\eta$ and the complex two-dimensional coordinates defined by

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}}(\xi_1 + \mathrm{i}\xi_2) = \frac{1}{\sqrt{2}}\xi \mathrm{e}^{\mathrm{i}\varphi_\xi} \\ z_2 &= \frac{1}{\sqrt{2}}(\eta_1 + \mathrm{i}\eta_2) = \frac{1}{\sqrt{2}}\eta \mathrm{e}^{\mathrm{i}\varphi_\eta}. \end{aligned} \tag{1}$$

Then the Kustaanheimo–Stiefel transformation takes the form

$$n_{\boldsymbol{x}}^{\mu} = r_0 \bar{z}_\alpha (\sigma^\mu)_{\alpha\beta} z_\beta \qquad \sigma^\mu = (1, \boldsymbol{\sigma}).$$

The Schrödinger equation for the 3D HA

$$\left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial \boldsymbol{x}^2} - \frac{e^2}{r}\right)\psi = E\psi \tag{2}$$

may be rewritten in the four-dimensional form as

$$\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi_\alpha\partial\xi_\alpha} + \frac{1}{2}m\omega^2\xi_\alpha\xi_\alpha\right) + (\xi \to \eta)\right]\psi = \varepsilon\psi \tag{3}$$

$$\left(\frac{\partial}{\partial\varphi_\xi} + \frac{\partial}{\partial\varphi_\eta}\right)\psi = 0 \tag{4}$$

where the following notations are introduced:

$$m = 4\mu \qquad \varepsilon = \frac{e^2}{2r_0} \qquad r_0 = \frac{\hbar^2}{\mu e^2} \qquad \omega = \left(-\frac{E}{2\mu}\right)^{1/2}.$$

Let a solution $\psi = \psi(\xi, \eta, \varphi_\xi, \varphi_\eta)$ of equations (3) and (4) be known, then we can obtain the solution of equation (2) setting $\varphi_\xi = -\varphi_\eta = \varphi/2$. The Kustaanheimo–Stiefel transformation reduces to the usual transformation to the parabolic coordinates

$$\begin{aligned} x_1 &= r_0\xi\eta\,\cos\varphi \qquad & x_2 &= r_0\xi\eta\,\sin\varphi \\ x_3 &= \frac{r_0}{2}(\xi^2 - \eta^2) \qquad & r &= \frac{r_0}{2}(\xi^2 + \eta^2). \end{aligned}$$

Let $E < 0$; then rescaling the coordinates we can reduce equations (3) and (4) to two Schrödinger equations for two 2D HO with unit mass and frequency and with the same values of angular momentum. The functions

$$\Psi_{n_1 n_2 m}(\boldsymbol{x}) = \psi_{n_1 n_2 m}\left(\frac{\boldsymbol{x}}{n r_0}\right)$$
$$n = n_1 + n_2 + |m| + 1 \qquad E = -\frac{\varepsilon}{n^2}$$

obey equation (2), where

$$\psi_{n_1 n_2 m}(\boldsymbol{x}) \equiv |n_1 n_2 m\rangle = (-1)^{n_1 + \frac{1}{2}(m - |m|)} \frac{\mathrm{e}^{\mathrm{i}m\phi}}{\pi^{1/2}} \varphi_{n_1|m|}(\xi)\varphi_{n_2|m|}(\eta)$$

and

$$\varphi_{n|m|}(\xi) = \mathrm{e}^{-\xi^2/2}\xi^{|m|}\left(\frac{n!}{(n+|m|)!}\right)^{1/2} L_n^{|m|}(\xi^2)$$

is a solution of the radial Schrödinger equation for the 2D HO with unit mass and frequency and the angular momentum equal to $|m|$. The above solutions are real and normalized as

$$\int_0^\infty \mathrm{d}(\xi^2)\,\varphi_{n|m|}(\xi)\varphi_{n'|m|}(\xi) = \delta_{nn'}.$$

Taking into account that $r^{-1}\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \mathrm{d}(\xi^2)\mathrm{d}(\eta^2)\mathrm{d}\varphi$ we obtain

$$\langle n_1 n_2 m \mid n'_1 n'_2 m'\rangle \equiv \int_{\mathbb{R}^3} r^{-1}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\psi_{n_1 n_2 m}(\boldsymbol{x})\psi_{n'_1 n'_2 m'}(\boldsymbol{x}) = \delta_{n_1 n'_1}\delta_{n_2 n'_2}\delta_{mm'}. \tag{5}$$

The measure in the right-hand side of the above expression is just a Lorentz-invariant measure on the light cone $n_{\boldsymbol{x}} \cdot n_{\boldsymbol{x}} = 0$.

2.2. The dynamical symmetry group

Let us introduce the operators a_α, b_α and their Hermitean conjugates as

$$\begin{aligned} a_\alpha &= \frac{1}{\sqrt{2}}(\bar{z}_\alpha + \partial_{z_\alpha}) \qquad & a_\alpha^\dagger &= \frac{1}{\sqrt{2}}(z_\alpha - \partial_{\bar{z}_\alpha}) \\ b_\alpha &= \frac{1}{\sqrt{2}}\varepsilon_{\alpha\beta}(z_\beta + \partial_{\bar{z}_\beta}) \qquad & b_\alpha^\dagger &= \frac{1}{\sqrt{2}}\varepsilon_{\alpha\beta}(\bar{z}_\beta - \partial_{z_\beta}). \end{aligned} \tag{6}$$

Then nonvanishing commutation relations are

$$[a_\alpha, a_\beta^\dagger] = [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}.$$

We can express the vectors $|n_1 n_2 m\rangle$ as

$$|n_1 n_2 m\rangle = [n_1!(n_1+|m|)!n_2!(n_2+|m|)!]^{-1/2} \times \begin{cases} a_1^{\dagger n_2+|m|} a_2^{\dagger n_1} b_1^{\dagger n_1+|m|} b_2^{\dagger n_2}|0\rangle & m \geqslant 0 \\ a_1^{\dagger n_2} a_2^{\dagger n_1+|m|} b_1^{\dagger n_1} b_2^{\dagger n_2+|m|}|0\rangle & m \leqslant 0 \end{cases} \tag{7}$$

where

$$|0\rangle = \exp(-z_\alpha \bar{z}_\alpha) \qquad a_\alpha|0\rangle = b_\alpha|0\rangle = 0 \qquad \alpha = 1, 2. \tag{8}$$

The linear shell of the vectors $|n_1 n_2 m\rangle$ may be considered as a subspace in the Fock space of a bosonic system of four degrees of freedom; this subspace is defined by the constraint

$$(a_\alpha a_\alpha^\dagger - b_\alpha b_\alpha^\dagger)|\mathrm{phys}\rangle = 0 \tag{9}$$

which, as can be readily seen, coincides with (4). We denote this subspace as H_{phys}.

From (7) it follows that the representation of the algebra so (4, 2)

$$\begin{array}{ll} L_{ij} = \frac{1}{2}(a^\dagger\sigma_k a + b^\dagger\sigma_k b) & L_{i5} = -\frac{1}{2}(a^\dagger\sigma_i C b^\dagger - aC\sigma_i b) \\ L_{i0} = \frac{1}{2\mathrm{i}}(a^\dagger\sigma_i C b^\dagger + aC\sigma_i b) & L_{50} = \frac{1}{2}(a^\dagger a + b^\dagger b + 2) \\ L_{i6} = -\frac{1}{2}(a^\dagger\sigma_i a - b^\dagger\sigma_i b) & L_{56} = \frac{\mathrm{i}}{2}(a^\dagger C b^\dagger - aCb) \\ L_{60} = \frac{1}{2}(a^\dagger C b^\dagger + aCb) & \end{array} \tag{10}$$

may be defined in H_{phys}, where $C = \mathrm{i}\sigma_2$ and the generators $L_{AB},\ A, B, \ldots = 0, \ldots, 3, 5, 6$ obey the commutation relations

$$[L_{AB}, L_{CD}] = \mathrm{i}(\eta_{AD}L_{BC} + \eta_{BC}L_{AD} - \eta_{AC}L_{BD} - \eta_{BD}L_{AC}) \tag{11}$$

where $\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, +1, -1)$. From (7) and (10) it follows that

$$L_{50}|n_1n_2m\rangle = (n_1 + n_2 + |m| + 1)|n_1n_2m\rangle. \tag{12}$$

Substituting (6) into (10) we obtain the following expressions for generators L_{AB} in configuration space:

$$\begin{array}{lll} L_{ij} = \varepsilon_{ijk}(\boldsymbol{x}\times\boldsymbol{p})_k & L_{i6} = -\frac{1}{2}x_i p^2 + p_i(\boldsymbol{xp}) + \frac{1}{2}x_i & \\ L_{i5} = -\frac{1}{2}x_i p^2 + p_i(\boldsymbol{xp}) - \frac{1}{2}x_i & L_{65} = (\boldsymbol{xp}) - \mathrm{i} & \\ L_{i0} = -rp_i & L_{60} = \frac{1}{2}(rp^2 - r) & L_{50} = \frac{1}{2}(rp^2 + r). \end{array} \tag{13}$$

The generators $L_{\mu\nu}$ induce the Lorentz transforms of four-vectors $n^\mu_{\boldsymbol{x}}$.

3. Discrete spectrum

3.1. The $Sp(2,\mathbf{R})/U(2)$ space

Let us consider a set of complex symmetric 2×2 matrices obeying the condition

$$I - \Lambda\Lambda^\dagger > 0. \tag{14}$$

On these matrices we can define the action of the $Sp(2,\mathbb{R})$ group, so that they become a symmetric space

$$SO(3,2)/(SO(3)\otimes SO(2)) \simeq Sp(2,\mathbb{R})/U(2).$$

This space has been considered in detail previously (see, e.g. [31]). Introduce a three-vector $\boldsymbol{u}$ as $\Lambda = C\boldsymbol{\sigma u}$. Then (14) is equivalent to the conditions

$$|\boldsymbol{u}^2| < 1 \qquad 1 - 2\boldsymbol{uu}^* + \boldsymbol{u}^2\boldsymbol{u}^{*2} > 0. \tag{15}$$

The infinitesimal operators corresponding to the action of the $SO(3,2)$ group on this space are given by [31]

$$\begin{array}{ll} L_{ij} = \mathrm{i}\left(u_i\dfrac{\partial}{\partial u_j} - u_j\dfrac{\partial}{\partial u_i}\right) & L_{50} = \boldsymbol{u}\dfrac{\partial}{\partial\boldsymbol{u}} \\ L_{5i} = \mathrm{i}\left(\dfrac{1+\boldsymbol{u}^2}{2}\dfrac{\partial}{\partial u_i} - u_i\left(\boldsymbol{u}\dfrac{\partial}{\partial\boldsymbol{u}}\right)\right) & \\ L_{0i} = -\left(\dfrac{1-\boldsymbol{u}^2}{2}\dfrac{\partial}{\partial u_i} + u_i\left(\boldsymbol{u}\dfrac{\partial}{\partial\boldsymbol{u}}\right)\right). & \end{array} \tag{16}$$

Let us introduce a unit complex four-vector as

$$k^\mu_{\boldsymbol{u}} = \left(\frac{1+\boldsymbol{u}^2}{1-\boldsymbol{u}^2}, \frac{2\boldsymbol{u}}{1-\boldsymbol{u}^2}\right) \qquad k_{\boldsymbol{u}}\cdot k_{\boldsymbol{u}} = 1. \tag{17}$$

Then we can rewrite conditions (15) as

$$w_{\boldsymbol{u}}^{0} > 0 \qquad w_{\boldsymbol{u}} \cdot w_{\boldsymbol{u}} = \frac{1 - 2\boldsymbol{u}\boldsymbol{u}^{*} + \boldsymbol{u}^{2}\boldsymbol{u}^{*2}}{|1 + \boldsymbol{u}^{2}|^{2}} > 0 \tag{18}$$

where $w_{\boldsymbol{u}}^{\mu} = \operatorname{Re} k_{\boldsymbol{u}}^{\mu}$. The action of generators $L_{\mu\nu}$ (16) corresponds to the Lorentz transforms of the vector $k_{\boldsymbol{u}}^{\mu}$.

3.2. Coherent states

Let $\boldsymbol{u}$ be a complex three-vector having the components

$$\boldsymbol{u} = \left(\tfrac{\mathrm{i}}{2}(\lambda_2 - \lambda_1), \tfrac{1}{2}(\lambda_1 + \lambda_2), 0\right) \tag{19}$$

and satisfying conditions (15). We now construct the superposition of states

$$|\boldsymbol{u}\rangle = c_0 \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\lambda_1\lambda_2)^{\frac{1}{2}(2n+|m|+1)} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} |nnm\rangle. \tag{20}$$

Using the formulas [32]

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) z^n$$
$$= (1-z)^{-1} \exp\left(-z\frac{x+y}{1-z}\right) (-xyz)^{-\alpha/2} J_{\alpha}\left(2\frac{(-xyz)^{1/2}}{1-z}\right) \qquad |z| < 1$$
$$\sum_{n=-\infty}^{\infty} t^n J_n(z) = \exp\left[(t - t^{-1})z/2\right] \tag{21}$$

we obtain

$$\langle \boldsymbol{x} \mid \boldsymbol{u}\rangle = \frac{c_0}{2\sqrt{\pi}} (\boldsymbol{k}_{\boldsymbol{u}}^{2})^{1/2} \exp(-k_{\boldsymbol{u}} \cdot n_{\boldsymbol{x}}). \tag{22}$$

It can readily be seen that an arbitrary three-vector satisfying conditions (15) can be obtained by applying the $SO(3)$ transformations to a certain three-vector defined by (19). Due to (13) such a transformation corresponds to certain transformation in H_{phys}. Then the vector $|\boldsymbol{u}\rangle$ defined by the right-hand side of equality (22) can be represented as a superposition of vectors of the H_{phys} space for an arbitrary $\boldsymbol{u}$ which obeys conditions (15). Then hereafter we will consider $\boldsymbol{u}$ as an arbitrary element of the space $Sp(2,\mathbb{R})/U(2)$.

Let us choose the normalization constant so that $\langle \boldsymbol{u} \mid \boldsymbol{u}\rangle = 1$, i.e.

$$|c_0|^2 = \frac{1 - 2\boldsymbol{u}\boldsymbol{u}^{*} + \boldsymbol{u}^{2}\boldsymbol{u}^{*2}}{|\boldsymbol{u}^{2}|}.$$

Thus, both conditions (15) are necessary; the first one is necessary for convergence of the series (20) and the second one, for normalizability of the resulting expression. Using (18) we finally obtain

$$\langle \boldsymbol{x} \mid \boldsymbol{u}\rangle = \frac{1}{\pi^{1/2}} (w_{\boldsymbol{u}} \cdot w_{\boldsymbol{u}})^{1/2} \exp(-k_{\boldsymbol{u}} \cdot n_{\boldsymbol{x}}). \tag{23}$$

CS for the 1D HA constructed in [9] may easily be obtained as a particular case of (23) putting $x^1 = x^2 = k_{\boldsymbol{u}}^1 = k_{\boldsymbol{u}}^2 = 0$.

3.3. Symmetry properties

Now we show that the system $|\boldsymbol{u}\rangle$ is $SO(3,2)$- invariant. This system is obviously $SO(3,1)$-invariant; on the other hand, we have the equality

$$\mathrm{e}^{\mathrm{i}\varepsilon L_{50}}|\boldsymbol{u}\rangle = \mathrm{e}^{\mathrm{i}\varepsilon}|\boldsymbol{u}\mathrm{e}^{\mathrm{i}\varepsilon}\rangle$$

which can be proved either using (12) and (20) or in the infinitesimal form using (13) and (16). Then the $SO(3,2)$-invariance of the system follows from the commutation relations (11).

From here it follows that the system $|\boldsymbol{u}\rangle$ is a system of the Perelomov's CS for the $SO(3,2)$ group constructed starting from the $SO(3)\times SO(2)$-invariant vector $|0\rangle$. This fact also can be proved directly for the following particular cases:

$$\Lambda = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \qquad \alpha_1\alpha_2 = 0. \tag{24}$$

To this end, let us introduce the operators A_α, B_α as

$$a_\alpha = A_\alpha + \mathrm{i}B_\alpha \qquad b_\alpha = A_\alpha - \mathrm{i}B_\alpha. \tag{25}$$

Since the matrices $C\sigma_i$ are symmetric, from (10) it follows that generators of the $SO(3,2)$ subgroup of the $SO(4,2)$ group may be represented as a nondegenerate linear combination of generators of the $Sp(2,\mathbb{R}) \simeq SO(3,2)$ group

$$X_{\alpha\beta} = A_\alpha A_\beta + B_\alpha B_\beta \qquad X^\dagger_{\alpha\beta} = A^\dagger_\alpha A^\dagger_\beta + B^\dagger_\alpha B^\dagger_\beta$$
$$Y_{\alpha\beta} = \tfrac{1}{2}(A_\alpha A^\dagger_\beta + A^\dagger_\beta A_\alpha) + \tfrac{1}{2}(B_\alpha B^\dagger_\beta + B^\dagger_\beta B_\alpha). \tag{26}$$

It follows from (26) that the $Sp(2,\mathbb{R})$ group is a group of canonical transformations of operators A_α, $A^\dagger_\alpha$ and B_α, $B^\dagger_\alpha$ separately. Since (8) is equivalent to

$$A_\alpha|0\rangle = B_\alpha|0\rangle = 0$$

we can use the analogy to the usual CS for a bosonic system of two degrees of freedom [16] to obtain the equalities

$$(A_\alpha - \Lambda_{\alpha\beta}A^\dagger_\beta)|\Lambda\rangle = (B_\alpha - \Lambda_{\alpha\beta}B^\dagger_\beta)|\Lambda\rangle = 0. \tag{27}$$

Denote as $\mathcal{H}_A$ the Hilbert space of states of bosonic system of two degrees of freedom composed by vectors of the form $A^\dagger_{\alpha_1}\cdots A^\dagger_{\alpha_n}|0\rangle_A$, where $A_\alpha|0\rangle_A = 0$, $\alpha = 1,2$. Following [16] define in $\mathcal{H}_A$ the CS system for the space $Sp(2,\mathbb{R})/U(2)$ as

$$|\Lambda\rangle_A = [\det(I - \Lambda\Lambda^\dagger)]^{1/4}\exp\left(\tfrac{1}{2}\Lambda_{\alpha\beta}A^\dagger_\alpha A^\dagger_\beta\right)|0\rangle_A.$$

The space $\mathcal{H}_B$ and its CS$|\Lambda\rangle_B$ may be defined analogously. Then we can consider the representation (26) of the $SO(3,2)$ group as acting in the subspace of the space $\mathcal{H}_A\times\mathcal{H}_B$ defined by the constraint (9). In the space $\mathcal{H}_A\times\mathcal{H}_B$, we consider the system of states

$$|\Lambda\rangle = |\Lambda\rangle_A \otimes |\Lambda\rangle_B = [\det(I - \Lambda\Lambda^\dagger)]^{1/2}\exp\left(\tfrac{1}{2}\Lambda_{\alpha\beta}X^\dagger_{\alpha\beta}\right)|0\rangle \tag{28}$$

where $X_{\alpha\beta}$ are given by (26). In view of (27), it is easy to show that the vectors $|\Lambda\rangle$ obey the constraint (9) and consequently belong to $\mathcal{H}_{\mathrm{phys}}$. From here it follows that (28) is a true Perelomov's HA CS system for the space $Sp(2,\mathbb{R})/U(2)$ and consequently it must coincide with (23). Using equality (7) and the formula from [32]

$$\sum_{n=0}^{\infty} L^\alpha_n(x)z^n = (1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right) \qquad |z| < 1$$

we can directly prove that (28) indeed coincides with (23) up to a phase factor if Λ has the form (24).

3.4. Robertson relations

Introduce the Hermitean operators $Q_a,\ a = 1, \ldots, 8$ as

$$Q_\alpha = \xi_\alpha \qquad Q_{\alpha+2} = -\mathrm{i}\frac{\partial}{\partial \xi_\alpha}$$
$$Q_{\alpha+4} = \eta_\alpha \qquad Q_{\alpha+6} = -\mathrm{i}\frac{\partial}{\partial \eta_\alpha}$$

and define their dispersion in a given state as

$$\Sigma_{ab} = \tfrac{1}{2}\langle Q_a Q_b + Q_b Q_a\rangle - \langle Q_a\rangle\langle Q_b\rangle.$$

By the virtue of (1), (6), (25) and (27), the operators Q_a acting on the vectors $|\boldsymbol{u}\rangle$ satisfy $4 = 8/2$ linearly independent equalities. The Robertson inequality for the dispersion of a set of Hermitean operators

$$\det\Sigma \geqslant \det\Omega \qquad \Omega_{ab} = -\tfrac{\mathrm{i}}{2}\langle[Q_a, Q_b]\rangle$$

in view of the results of [33], is then transformed into equality, if the mean values are taken in an arbitrary CS $|\boldsymbol{u}\rangle$.

4. Continuous spectrum

4.1. The SO(4, 1) group

Let us introduce the generators

$$\boldsymbol{\Pi}_i^{\pm} = L_{6i} \pm L_{0i}.$$

The generators $\boldsymbol{\Pi}^+$ and $\boldsymbol{\Pi}^-$ form two Abelian subgroups, which we denote by $\mathcal{T}^+$ and $\mathcal{T}^-$; we denote the subgroups induced by the generators L_{06} and L_{ij} as $\mathcal{T}^0$ and $\mathcal{R}$, respectively. Finite transformations are denoted as

$$\Theta_{\pm}(\boldsymbol{a}) = \exp(\mathrm{i}\boldsymbol{\Pi}^{\pm}\boldsymbol{a}) \qquad \Theta_0(\varepsilon) = \exp(\mathrm{i}L_{06}\varepsilon).$$

Consider the action $\boldsymbol{v} \mapsto \boldsymbol{v}_g$ of elements $g \in SO(4,1)$ on the vectors $\boldsymbol{v} \in \mathbb{R}^3$ defined by

$$\begin{aligned} &g = \Theta_-(\boldsymbol{a})\colon \boldsymbol{v}_g = \boldsymbol{v} - \boldsymbol{a} \\ &g = \Theta_+(\boldsymbol{a})\colon \boldsymbol{v}_g = \frac{\boldsymbol{v} + \boldsymbol{a}v^2}{1 + 2\boldsymbol{v}\boldsymbol{a} + v^2 a^2} \\ &g = \Theta_0(\varepsilon)\colon \boldsymbol{v}_g = \boldsymbol{v}\mathrm{e}^{\varepsilon}. \end{aligned} \tag{29}$$

The generators have the form

$$\begin{aligned} &\mathrm{i}\boldsymbol{\Pi}^- = -\frac{\partial}{\partial \boldsymbol{v}} \qquad \mathrm{i}\boldsymbol{\Pi}^+ = v^2\frac{\partial}{\partial \boldsymbol{v}} - 2\boldsymbol{v}\left(\boldsymbol{v}\frac{\partial}{\partial \boldsymbol{v}}\right) \\ &\mathrm{i}L_{06} = \boldsymbol{v}\frac{\partial}{\partial \boldsymbol{v}} \qquad \mathrm{i}L_{ik} = v_k\frac{\partial}{\partial v_i} - v_i\frac{\partial}{\partial v_k}. \end{aligned} \tag{30}$$

The stationary subgroup of the point $\boldsymbol{v} = \mathbf{o}$ is $\mathcal{K} = \mathcal{T}^+ \circledS (\mathcal{T}^0 \otimes \mathcal{R})$; then the space $\mathbb{R}^3$ equipped with such an action of the $SO(4,1)$ group may be identified with the coset space $SO(4,1)/\mathcal{K}$. Then the action of the $SO(4,1)$ group on the unit real four-vector k_ν (17) is defined; the generators $L_{\mu\nu}$ correspond to the Lorentz transforms of this vector.

4.2. Wavefunctions

In the case of positive energy, we can use the coordinate rescaling to reduce equations (3) and (4) to the Schrödinger equations for two 'oscillators' having unit mass, frequency equal to i, and the same values of angular momentum. Then the solutions of equation (2) corresponding to energy $E = \varepsilon(\rho_1 + \rho_2)^{-2}$ are

$$\Psi_{\rho_1\rho_2 m}(\boldsymbol{x}) = \psi_{\rho_1\rho_2 m}(\boldsymbol{x}(r_0(\rho_1 + \rho_2))^{-1})$$

where

$$\psi_{\rho_1\rho_2 m}(\boldsymbol{x}) \equiv |\rho_1\rho_2 m\rangle = \mathrm{e}^{\mathrm{i}m\phi}\varphi_{\rho_1|m|}(\xi)\varphi_{\rho_2|m|}(\eta)$$

$$\varphi_{\rho|m|}(\xi) = (2\pi\mathrm{i}\mathrm{e}^{\pi\rho})^{-1/2}\left|\Gamma\left(-\mathrm{i}\rho + \frac{|m|+1}{2}\right)\right| \times \mathrm{e}^{\mathrm{i}\xi^2/2}(-\mathrm{i}\xi^2)^{|m|/2}\,{}_1F_1\left(-\mathrm{i}\rho + \frac{|m|+1}{2}, |m|+1, -\mathrm{i}\xi^2\right).$$

Within the change $|m|$ to $2l + 1$, the functions $\varphi_{\rho|m|}$ coincide with radial components of wavefunctions of the continuous spectrum of HA obtained in [34]. The normalization factors are chosen so that the functions $\varphi_{\rho|m|}(\xi)$ are real and satisfy the normalization conditions

$$\int_0^\infty \mathrm{d}(\xi^2)\,\varphi_{\rho|m|}(\xi)\varphi_{\rho'|m|}(\xi) = \delta(\rho - \rho').$$

Then

$$\langle\rho_1\rho_2 m|\rho'_1\rho'_2 m'\rangle = \delta(\rho_1 - \rho'_1)\delta(\rho_2 - \rho'_2)\delta_{mm'}.$$

The equality

$$L_{06}|\rho_1\rho_2 m\rangle = -(\rho_1 + \rho_2)|\rho_1\rho_2 m\rangle \tag{31}$$

holds.

4.3. Coherent states

Let $\boldsymbol{v} = (-v\ \cos\theta, v\ \sin\theta, 0)$. By the analogy with (20), define the states $|\boldsymbol{v}\rangle$ as

$$|\boldsymbol{v}\rangle = \left(\boldsymbol{k}_v^2\right)^{-1/2}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\theta}\int_{-\infty}^{\infty}\mathrm{d}\rho\, v^{-2\mathrm{i}\rho}|\rho\rho m\rangle. \tag{32}$$

Inverting the Mellin transform of the Bessel function [35]

$$\int_0^\infty t^{s-1}\,\mathrm{d}t\,(1+t^2)^{-1}\exp\left(-\mathrm{i}\frac{\xi^2+\eta^2}{2}\frac{1-t^2}{1+t^2}\right)J_{|m|}\left(-\mathrm{i}\xi\eta\frac{2t}{1+t^2}\right) = \pi\,\mathrm{i}^{|m|+1}\mathrm{e}^{\pi\rho}\,\varphi_{\rho|m|}(\xi)\varphi_{\rho|m|}(\eta)\big|_{\mathrm{i}\rho=\frac{s-1}{2}}$$

and, in view of (21), we obtain

$$\langle\boldsymbol{x}|\boldsymbol{v}\rangle = \exp(-\mathrm{i}k_{\boldsymbol{v}}\cdot n_{\boldsymbol{x}}). \tag{33}$$

Using (31) and (32) we obtain that at $g \in \mathcal{T}^0$

$$T(g)|\boldsymbol{v}\rangle = \left(\frac{\mathrm{d}\mu\left(k_{\boldsymbol{v}_g}\right)}{\mathrm{d}\mu(k_{\boldsymbol{v}})}\right)^{1/3}|\boldsymbol{v}_g\rangle \tag{34}$$

where $T(g)$ is a representation of the $SO(4,1)$ group with the Lie algebra given by (13) and $\mathrm{d}\mu(k) = (k^0)^{-1}\,\mathrm{d}^3\boldsymbol{k}$ is the Lorentz-invariant measure on the hyperboloid $k\cdot k = 1$. Equality

(34) can also be proved in the infinitesimal form using (13) and (30). On the other hand, the validity of (34) at $g \in SO(3, 1)$ is obvious, so it is correct for all $g \in SO(4, 1)$.

Define the space $SO(4, 1)/(\mathcal{T}^{+}Ⓢ\mathcal{R})$ as a set of pairs $(\boldsymbol{v}, \tau)$, where $\tau \in \mathbb{R}\backslash\{0\}$ and the action of the $SO(4, 1)$ group is defined by (29) and

$$\tau_g = \left(\frac{\mathrm{d}\mu(k_{v_g})}{\mathrm{d}\mu(k_\epsilon)}\right)^{1/3} \tau.$$

Then the states

$$|\boldsymbol{v}\tau\rangle = \tau^{-1}|\boldsymbol{v}\rangle$$

compose a system of the Perelomov's CS for the mentioned space.

5. Relation to the conformal group

The twistor space $SO(4, 2)/(SO(4) \otimes SO(2))$ is a domain in $\mathbb{C}^4$ defined by the inequalities

$$|u_a u_a| < 1 \qquad 1 - 2u_a u_a^* + |u_a u_a|^2 > 0 \qquad a = 1, \ldots, 4. \tag{35}$$

We obtain another realization of the twistor space considering the mapping

$$z^0 = \mathrm{i}\frac{1 + u_a u_a - 2u_4}{1 - u_a u_a + 2\mathrm{i}u_4} \qquad z^k = \frac{2\mathrm{i}u_k}{1 - u_a u_a + 2\mathrm{i}u_4}. \tag{36}$$

Then (35) transforms to

$$\Im z^0 > 0 \qquad \Im z \cdot \Im z > 0.$$

Consider the set of all holomorphic C^∞-functions which are square integrable over the twistor space with respect to the measure $d^4 z d^4 \bar{z}$. On this set, we can define the $SO(4, 2)$ group irreducible representation belonging to the discrete series and having the generators

$$\begin{aligned}
&\mathrm{i}(L_{5\mu} + L_{6\mu}) = (z \cdot z)\frac{\partial}{\partial z^\mu} - 2z_\mu \left(z \cdot \frac{\partial}{\partial z}\right) - 2z_\mu \\
&\mathrm{i}(L_{5\mu} - L_{6\mu}) = \frac{\partial}{\partial z^\mu} \\
&\mathrm{i}L_{\mu\nu} = z_\mu \frac{\partial}{\partial z^\nu} - z_\nu \frac{\partial}{\partial z^\mu} \qquad \mathrm{i}L_{65} = z \cdot \frac{\partial}{\partial z} + 1.
\end{aligned} \tag{37}$$

Consider the functions

$$\langle \boldsymbol{x} \mid z\rangle = \mathrm{e}^{\mathrm{i}n_x \cdot z}.$$

Then one can show [36] that the integral transform

$$F(z) = \int \frac{\mathrm{d}^3\boldsymbol{x}}{r} \langle \boldsymbol{x} \mid z\rangle f(\boldsymbol{x}) \tag{38}$$

intertwines the representations (13) and (37) of the $SO(4, 2)$ group. At the level of Lie algebras, this fact can be directly observed since the difference between generators (13) and (37) vanishes acting on the functions $\langle \boldsymbol{x} \mid z\rangle$.

We can pass from the twistor space to the $SO(3, 2)/SO(3) \times SO(2)$ space letting $u_4 = 0$; then (35) transforms into (15), and from (36) it follows that $z^\mu = \mathrm{i}k_{\boldsymbol{u}}^\mu$. On the other hand, the functions $\langle \boldsymbol{x}|z\rangle$ transform into CS given by (23) up to a normalization factor.

Letting $\Im z^\mu = 0$ we pass to the Shilov boundary of the twistor space which coincides with the Minkowski space. If we additionally let $z^\mu = k_v^\mu$ then the functions $\langle \boldsymbol{x} \mid z\rangle$ pass into the states (33).

Passing to the Shilov boundary, representation (37) transforms into the representation which describes massless spin-zero particles over the Minkowski space [37]. In this case, the transform (38) shows the coincidence of representations of the $SO(4, 2)$ group which describe the hydrogen atom and massless, spin-zero particles on the Minkowski space. This coincidence has been proved in a more complicated way in [38].

Let us consider now the manifold which belongs to the boundary of the twistor space and is defined by the equality $\Im z \cdot \Im z = 0$ (however, we still have $\Im z^0 > 0$) and, moreover, we assume that $z_i z_i = 0$. Then we can represent the HA wavefunctions in the form of an integral over this manifold of the functions $\langle \boldsymbol{x}|z\rangle$ with a certain weight factor [30]. This indicates the possibility of a quasi-classical description of HA in terms of CS constructed above. This problem requires a further investigation.

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